

# On relation between geometric momentum and annihilation operators on a two-dimensional sphere

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With a recently introduced *geometric momentum* that depends on the extrinsic curvature and offers a proper description of momentum on two-dimensional sphere, we show that the annihilation operators whose eigenstates are coherent states on the sphere take the expected form  $\alpha\mathbf{x} + i\beta\mathbf{p}$ , where  $\alpha$  and  $\beta$  are two operators that depend on the angular momentum and  $\mathbf{x}$  and  $\mathbf{p}$  are the position and the *geometric momentum*, respectively. Since the geometric momentum is manifestly a consequence of embedding the two-dimensional sphere in the three-dimensional flat space, the coherent states reflects some aspects beyond the intrinsic geometry of the surfaces.

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The coherent states on two-dimensional sphere, generated by the annihilation operators, were discovered around the turn of present century, independently by Hall [1–6] in the Bargmann representation, and by Kowalski and Rembieliński [7–10] in the position representation, respectively. Once each group of them got to know the work of the other, both soon realized that their coherent states are essentially the same [6, 9], and the equivalence was also noted by other group [11]. However, it is puzzling that in the annihilation operators they introduced, there is a fundamental quantity that is represented by a non-hermitian operator and has the same dimension of linear momentum, but it does not bear a transparent physical nor geometric meaning. This article points out that the physical and geometric interpretation of the fundamental quantity is easily available, based on the *geometric momentum* that is recently introduced to offer a proper description of momentum on sphere [12], whose general form for an arbitrary two-dimensional curved surface with  $M$  denoting the mean curvature, is given by [12–14],

$$\mathbf{p} \equiv -i\hbar(\mathbf{r}^\mu \partial_\mu + M\mathbf{n}), \quad (1)$$

where  $\mathbf{r} = [x(x^\mu, x^\nu), y(x^\mu, x^\nu), z(x^\mu, x^\nu)]$  is the position vector on the surface,  $\mathbf{r}^\mu = g^{\mu\nu}\mathbf{r}_\nu = g^{\mu\nu}\partial\mathbf{r}/x^\nu$ , and at this point  $\mathbf{r}$ ,  $\mathbf{n} = (n_x, n_y, n_z)$  denotes the normal vector and  $M\mathbf{n}$  symbolizes the mean curvature vector field, a geometric invariant [12]. When first seeing this expression (1) that apparently contains a term  $M\mathbf{n}$ , many thinks that it has component along the normal direction  $\mathbf{n}$ . It is in fact an operator exclusively defined on the tangent plane to the surface at the given point for we have an operator relation  $\mathbf{p} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{p} = 0$  with use of a relation  $\nabla_2 \cdot \mathbf{n} = -2M$  [15]. In flat space with  $M = 0$ , operator  $\mathbf{p}$  assumes its usual form  $-i\hbar\nabla$ . Noting that the mean curvature  $M$  is an extrinsic curvature, the momentum (1) actually reflects a consequence of embedding the two-dimensional surface in three-dimensional flat space. Other theoretical and experimental progresses relating the quantum mechanics to the extrinsic curvature, please refer to papers [16–20]. Why this momentum (1) is advantageous over other canonical one comes from three respects: i) According to Dirac, the commutator between the canonical momentum (1) and the Hamiltonian of the system under consideration must take the same form as the classical Poisson bracket for these two quantities, especially "for systems for which the orders of the factors of products occurring in the equations of motion are unimportant" [21]. As we showed in paper [13] for the spherical surface, the canonical momentum  $p_\theta$  violates this rule whereas the geometric one is completely compatible with it. ii) From the point of the Dirac theory for systems with second class constraints, the geometric momentum is nothing but the direct canonical quantization of the tangential momentum on the surface *provided that the constraints are formulated in the phase-space rather than the configuration one* [13, 22]. iii) For the two-dimensional sphere of unit radius, the momentum (1) possesses a complete set of eigenfunctions [13, 23] and have in principle observable consequence [24]. In contrast, the canonical operator  $p_\theta$  is not a self-adjoint one.

Though Hall's formalism is quite generally applicable, e.g., easily to be generalized for sphere in any dimensions [6], while that of Kowalski and Rembieliński has advantages to get the explicit form of the coherent states on some compact manifolds, e.g., on the torus [10], the present paper is limited within the two-dimensional sphere. The

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coherent states are labeled by points in the associated classical phase space, namely the cotangent bundle  $T^*S^2 = \{(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x^2 = a^2, \mathbf{x} \cdot \mathbf{p} = 0\}$  where  $a$  is the radius of the sphere and both position  $\mathbf{x}$  and momentum  $\mathbf{p}$  are defined for a single particle on the sphere. In quantum mechanics, the relation  $\mathbf{x} \cdot \mathbf{p} = 0$  must be replaced by the operator identity  $\mathbf{p} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{p} = 0$ .

In classical mechanics, the annihilation operators  $\mathbf{a}$  on the sphere are easily constructed via classical angular momentum  $j$  together with  $\mathbf{x}$  and  $\mathbf{p}$  as [6, 9],

$$\mathbf{a} = \cosh\left(\frac{j}{m\omega r^2}\right) \mathbf{x} + i \frac{r^2}{j} \sinh\left(\frac{j}{m\omega r^2}\right) \mathbf{p}. \quad (2)$$

In quantum mechanics, with a "shifted, dimensionless angular momentum" [25] operator  $S$  defined by,

$$S \equiv \sqrt{J^2/\hbar^2 + 1/4}, \quad (3)$$

with  $\mathbf{J}$  being the angular momentum, the annihilation operators may take the following dimensionless form [6, 9],

$$\mathbf{Z} = \exp\left(\frac{1}{2}\eta\right) \left( \left[ \cosh(\eta S) + \frac{1}{2S} \sinh(\eta S) \right] \frac{\mathbf{X}}{a} + \frac{1}{S} i \sinh(\eta S) \mathbf{P} \right) \quad (4)$$

where  $\mathbf{X}$  is the corresponding operator of Cartesian positions  $\mathbf{x}$  on the sphere, and  $\mathbf{P} \equiv \mathbf{J}/\hbar \times \mathbf{X}/a$  is a non-hermitian operator of quantity that looks like a momentum, and  $\delta \equiv 1$  from Kowalski and Rembieliński [9] and  $\eta \equiv \hbar/m\omega a^2$  from Hall [6] who also argued why this form is necessary. The operator  $S$  and the angular momentum possess simultaneous eigenstates  $|jm\rangle$ , so the eigenvalues of  $S$  are simply,

$$S |jm\rangle = \left(j + \frac{1}{2}\right) |jm\rangle. \quad (5)$$

First, we are going to address the following issues: The direct correspondence of  $\mathbf{P}$  in (4) and  $\mathbf{p}$  in (2) is not accurate for we can easily show that the relation  $\mathbf{P} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{P} = 0$  does not hold. Thus we have to rewrite  $\mathbf{P}$  in (4) in terms of the geometric momentum (1) that has exact classical correspondence.

After symmetrization of the  $\mathbf{P}$ , we have its corresponding Hermitian operator  $\mathbf{\Pi}$ ,

$$\mathbf{\Pi} \equiv \frac{1}{2\hbar} \left( \mathbf{J} \times \frac{\mathbf{X}}{a} - \frac{\mathbf{X}}{a} \times \mathbf{J} \right). \quad (6)$$

On the sphere parametrized by,  $(0 < \theta < \pi, 0 \leq \varphi \leq 2\pi)$

$$x = a \sin \theta \cos \varphi, \quad y = a \sin \theta \sin \varphi, \quad z = a \cos \theta, \quad (7)$$

three components of the operator  $\mathbf{\Pi}$  are respectively,

$$\Pi_x = -i(\cos \theta \cos \varphi \frac{\partial}{\partial \theta} - \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \varphi} - \sin \theta \cos \varphi), \quad (8)$$

$$\Pi_y = -i(\cos \theta \sin \varphi \frac{\partial}{\partial \theta} + \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \varphi} - \sin \theta \sin \varphi), \quad (9)$$

$$\Pi_z = -i(-\sin \theta \frac{\partial}{\partial \theta} - \cos \theta). \quad (10)$$

We have also,

$$\mathbf{\Pi}^2 = \frac{J^2}{\hbar^2} + 1. \quad (11)$$

Rewriting  $\mathbf{\Pi}$  as

$$\mathbf{p} = \frac{\hbar}{a} \mathbf{\Pi}, \quad (12)$$

we see that  $\mathbf{\Pi}$  is nothing but the geometric momentum (1) on the the two-dimensional sphere [12, 13, 26], despite a constant factor.

Noting that the ratio  $\mathbf{X}/a$  is the unit direction operator  $\mathbf{N}$ ,

$$\mathbf{N} \equiv \frac{\mathbf{X}}{a}, \quad (13)$$

we see immediately how the annihilation operators (4) depend on  $S$ ,  $\mathbf{N}$ , and  $\mathbf{p}$  as,

$$\mathbf{Z} = \exp\left(\frac{1}{2}\eta\right) \left( \left[ \cosh(\eta S) - \frac{1}{2S} \sinh(\eta S) \right] \mathbf{N} + \frac{1}{S} i \sinh(\eta S) \mathbf{\Pi} \right). \quad (14)$$

With help of a series of relations, e.g.,

$$\mathbf{N} \times \frac{\mathbf{J}}{\hbar} = -\mathbf{\Pi} + i\mathbf{N}, \quad \frac{\mathbf{J}}{\hbar} \times \mathbf{N} = \mathbf{\Pi} + i\mathbf{N}, \quad (15)$$

$$\mathbf{\Pi} \times \frac{\mathbf{J}}{\hbar} = \mathbf{N}(\mathbf{\Pi}^2 + 1), \quad \frac{\mathbf{J}}{\hbar} \times \mathbf{\Pi} = (-\mathbf{\Pi}^2 + 1)\mathbf{N}, \quad (16)$$

the following properties of operator (14) can be easily proved. i) The annihilation operators  $\mathbf{Z}$  is a vector operator that satisfies,

$$[J_i, Z_j] = i\varepsilon_{ijk} Z_k. \quad (17)$$

ii) The annihilation operators take following well-known form,

$$\mathbf{Z} = \alpha(J)\mathbf{N} + i\beta(J)\mathbf{\Pi} = \alpha(J)\frac{\mathbf{X}}{a} + i\beta(J)\frac{a\mathbf{p}}{\hbar}, \quad (18)$$

where  $\alpha(J)$  and  $\beta(J)$  are two Hermitian operators solely depending on the angular momentum  $J$ , and it formally coincides with the classical one (2) with considering the operator ordering.

*Secondly*, we give the Schwinger boson representation of the annihilation operators (18).

The Schwinger bosons are two sets of annihilation and creation operators  $\vec{a} \equiv (a_1, a_2)$  and  $\vec{a}^\dagger \equiv (a_1^\dagger, a_2^\dagger)$  [27] which satisfy the bosonic commutation relation  $[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$  with  $\alpha, \beta = 1, 2$ . The angular momentum operators are constructed out of Schwinger bosons as: [27]

$$J_k \equiv a^\dagger \frac{\sigma_k}{2} a, \quad (19)$$

where  $\sigma_k$  ( $k = 1, 2, 3$ ) denote the  $k$ -component of the Pauli matrices. It is easy to check that the operators in (19) satisfy the  $SU(2)$  Lie algebra with the  $SU(2)$  Casimir operator:

$$J^2 \equiv \frac{\vec{a}^\dagger \cdot \vec{a}}{2} \left( \frac{\vec{a}^\dagger \cdot \vec{a}}{2} + 1 \right). \quad (20)$$

Thus the angular momentum quantum number and operator  $S$  (3) can be expressed by the eigenvalues of total occupation number operator  $n \equiv n_1 + n_2$ ,

$$j = \frac{(n_1 + n_2)}{2} \equiv \frac{n}{2}, \text{ and } S = \frac{n+1}{2}, \quad (21)$$

where the occupation numbers are respectively  $n_1 = a_1^\dagger a_1$  and  $n_2 = a_2^\dagger a_2$ , and the eigenvalues of the occupation number operator are  $n_\alpha = 0, 1, 2, \dots$ . The unit direction vector  $\mathbf{N}$  and the geometric momentum  $\mathbf{\Pi}$  are respectively [12, 28, 29],

$$\mathbf{N} = \frac{1}{2} \left( \frac{1}{\sqrt{S(S+1)}} \mathbf{A} + \mathbf{A}^\dagger \frac{1}{\sqrt{S(S+1)}} \right), \quad (22)$$

$$\mathbf{\Pi} = \frac{1}{2} \left( \sqrt{S} \mathbf{B} \frac{1}{\sqrt{S}} + \frac{1}{\sqrt{S}} \mathbf{B} \sqrt{S} \right), \quad (23)$$

where,

$$\mathbf{A} = \left( \frac{1}{2} (a_2 a_2 - a_1 a_1), -\frac{i}{2} (a_2 a_2 + a_1 a_1), a_2 a_1 \right), \quad (24)$$

$$\mathbf{B} = \frac{i}{2} (\mathbf{A}^\dagger - \mathbf{A}). \quad (25)$$

Thus, we have the Schwinger boson representation of the annihilation operators (14)

$$Z_+ \equiv Z_x + iZ_y = \frac{\exp(1/2)}{2} \left( \frac{\exp(S)}{\sqrt{S(S+1)}} a_2 a_2 - \frac{\exp(-S)}{\sqrt{S(S-1)}} a_1^\dagger a_1^\dagger \right), \quad (26)$$

$$Z_- \equiv Z_x - iZ_y = \frac{\exp(1/2)}{2} \left( -\frac{\exp(S)}{\sqrt{S(S+1)}} a_1 a_1 + \frac{\exp(-S)}{\sqrt{S(S-1)}} a_2^\dagger a_2^\dagger \right), \quad (27)$$

$$Z_z = \frac{\exp(1/2)}{2} \left( \frac{\exp(S)}{\sqrt{S(S+1)}} a_2 a_1 + \frac{\exp(-S)}{\sqrt{S(S-1)}} a_2^\dagger a_1^\dagger \right). \quad (28)$$

From these operators (26)-(28), we can easily verify the following relations:

$$[Z_i, Z_j] = 0, \quad Z^2 = 1. \quad (29)$$

To conclude and remark, we see that the annihilation operators whose eigenstates are coherent states on a two-dimensional sphere turn out to be of the expected form  $\alpha \mathbf{x} + i\beta \mathbf{p}$ , where  $\alpha$ ,  $\beta$  and  $\mathbf{x}$ ,  $\mathbf{p}$  are four Hermitian operators, and as expected, the annihilation operators are expressible in the Schwinger boson representation. Moreover, we make a general conjecture that on an arbitrary surface the coherent states generated by the annihilation operators, once they exist, the similar relation (18) holds true between the annihilation operators and the geometric momentum. For two-dimensional surfaces, the mean curvature is an extrinsic one while the gaussian curvature is an intrinsic geometric property. So, the geometric momentum is not obtainable from the intrinsic point of view. Since the geometric momentum is manifestly a consequence of embedding the two-dimensional sphere in the three-dimensional flat space, the coherent states reflects some aspects of the extrinsic geometric properties of the surfaces.

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- [1] B. C. Hall, *J. Funct. Anal.* **122**(1994)103.
  - [2] B. C. Hall, *Comm. Math. Phys.* **184**(1997)233.
  - [3] B. C. Hall, *J. Funct. Anal.* **143**(1997)98.
  - [4] B. C. Hall, *Quantum mechanics in phase space*, in *Perspectives on quantization* (MA: South Hadley, 1996) pp.47–62.
  - [5] B. C. Hall, *Bull. (N.S.) Amer. Math. Soc.* **38**(2001)43.
  - [6] B. C. Hall, J. J. Mitchell, *J. Math. Phys.* **43**(2002)1211.
  - [7] K. Kowalski and J. Rembieliński, *J. Phys. A: Math. Theor.* **33**(2000)6035.
  - [8] K. Kowalski and J. Rembieliński, *J. Math. Phys.* **42**(2001)4138.
  - [9] K. Kowalski and J. Rembieliński, *J. Phys. A: Math. Theor.* **41**(2008)304021.
  - [10] K. Kowalski and J. Rembieliński, *Phys. Rev. A* **75**(2007)052102.
  - [11] C. Villegas-Blas, *J. Math. Phys.* **43**(2002)2249.
  - [12] Q. H. Liu, C. L. Tong, M. M. Lai, *J. Phys. A: Math. and Theor.* **40**(2007)4161.
  - [13] Q. H. Liu, L. H. Tang, D. M. Xun, *Phys. Rev. A* **84**(2011)042101.
  - [14] Q. H. Liu, arXiv:1109.0153v2, 4 Sep. 2011.
  - [15] C. E. Weatherburn, *Differential Geometry of Three Dimensions*, Vol. 1. (Cambridge University Press, 1930).
  - [16] H. Jensen and H. Koppe, *Ann. Phys. (N.Y.)* **63**(1971)586.
  - [17] R. C. T. da Costa, *Phys. Rev. A* **23**(1981)1982.
  - [18] G. Ferrari and G. Cuoghi, *Phys. Rev. Lett.* **100**(2008)230403.
  - [19] A. Szameit, *et. al*, *Phys. Rev. Lett.* **104**(2010)150403.
  - [20] J. Onoe, T. Ito, H. Shima, H. Yoshioka and S. Kimura, *Europhys. Lett.* **98**(2012)27001.
  - [21] P. A. M. Dirac, *Proc. R. Soc. Lond. A* **109**(1925)642.
  - [22] T. Homma, T. Inamoto, T. Miyazaki, *Phys. Rev. D* **42**(1990)2049.
  - [23] H. R. Sun, D. M. Xun, L. H. Tang and Q. H. Liu, *Commun. Theor. Phys.* **58**(2012)31.
  - [24] Q. H. Liu, arXiv:1209.2209v2, 1 Nov. 2012.
  - [25] B. C. Hall, J. J. Mitchell, arXiv:1112.1443v1, 6 Dec. 2011.
  - [26] D. M. Xun and Q. H. Liu, *Int. J. Geom. Meth. Mod. Phys.*, **10**(2013)1220031.
  - [27] J. Schwinger, *US Atomic Energy Commission*, Report NYO-3071,1952; later published in *Quantum Theory of Angular Momentum*, edited by L. C. Biendenharn and H. Van Dam (New York: Academic Press, 1965).

- [28] G. Gyorgyi, S. Kovesi-Domokos, *IL Nuovo Cimento B* **58**(1968)191.
- [29] X. Yang, D. M. Xun, X. P. Rong, L. Shan, Q. H. Liu, *Commun. Theor. Phys.* **57**(2012)575.